The Binomial Theorem

Given $n, r \in W$, and the following definition from Combinatorial mathematics,

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

where $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$, we can state the **Binomial Theorem** as,

$$\begin{aligned}
(p+q)^n &= \sum_{r=0}^{n} \binom{n}{r} p^{n-r} q^r \\
&= \binom{n}{0} p^{n-0} q^0 + \binom{n}{1} p^{n-1} q^1 + \binom{n}{2} p^{n-2} q^2 + \ldots + \binom{n}{n} p^{n-n} q^n \\
&= \frac{n!}{n!0!} p^n + \frac{n!}{(n-1)!1!} p^{n-1} q + \frac{n!}{(n-2)!2!} p^{n-2} q^2 + \ldots + \frac{n!}{0!n!} q^n \\
&= p^n + np^{n-1} q + \frac{n(n-1)}{2} p^{n-2} q^2 + \ldots + q^n
\end{aligned}$$

Examples

a)

$$\begin{aligned}
(3x+2)^4 &= \sum_{r=0}^{4} \binom{4}{r} (3x)^{4-r} 2^r \\
&= \binom{4}{0} (3x)^{4-0} 2^0 + \binom{4}{1} (3x)^{4-1} 2^1 + \binom{4}{2} (3x)^{4-2} 2^2 + \ldots + \binom{4}{4} (3x)^{4-4} 2^4 \\
&= 81x^4 + 216x^3 + 216x^2 + 96x + 16
\end{aligned}$$

b)

$$\begin{aligned}
(x - \frac{1}{x})^3 &= \sum_{r=0}^{3} \binom{3}{r} x^{3-r} (-\frac{1}{x})^r \\
&= \binom{3}{0} x^{3-0} (-\frac{1}{x})^0 + \binom{3}{1} x^{3-1} (-\frac{1}{x})^1 + \binom{3}{2} x^{3-2} (-\frac{1}{x})^2 + \binom{3}{3} x^{3-3} (-\frac{1}{x})^3 \\
&= x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}
\end{aligned}$$

Application: Probability

A Binomial Experiment is one in which we define two outcomes - one deemed a *success* which denote by the letter, p, and the other a *failure*, denoted by q. Tossing a coin is an obvious example. Since each outcome, a head or a tail is equally likely, or will 50% of the time, we make the assignments, $p = q = \frac{1}{2}$. If we perform the experiment n times, the Binomial Theorem can supply us with a complete probability distribution. Consider a coin tossed three times. Here is the probability distribution,

$$\begin{aligned}
(p+q)^3 &= p^3 + 3p^2 q + 3pq^2 + q^3 \\
\left(\frac{1}{2} + \frac{1}{2}\right)^3 &= \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 \\
&= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}
\end{aligned}$$