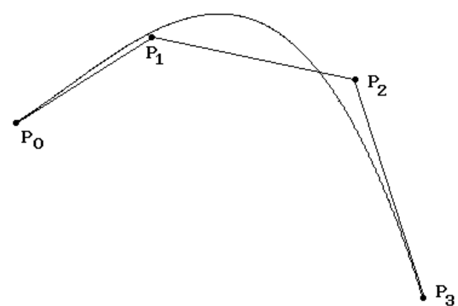
Bezier Curves

While working for the Renault automobile company in France, an engineer by the name of P. Bezier developed a system for designing car bodies based partly on some fairly straightforward mathematics. Although such a design would naturally take place in three dimensions, a two dimensional discussion will permit the reader to glimpse the power of Bezier's concept.

A (cubic) **Bezier curve** is completely determined by four consecutive points:

$$P_0(x_0, y_0), P_1(x_1, y_1), P_2(x_2, y_2), P_3(x_3, y_3)$$



The goal is to determine a third degree polynomial of the form,

$$y = ax^3 + bx^2 + cx + d \tag{1}$$

which *"fits"* these points according to the following criteria,

a) The polynomial is to pass through the endpoints, P_0 and P_3, and,
b) The slope at the endpoints is to agree with the slope determined by the endpoint and the adjacent point.

The polynomial does not pass through the two intermediary points, P_1 and P_2, called *control points* but rather their role is to influence the shape of the curve passing through P_0 and P_3.

Example

Suppose we wish to construct a Bezier curve through the points,

$$P_0(-3, 1), P_1(-1, 3)), P_2(2, 2) \text{ and } P_3(3, -3)$$

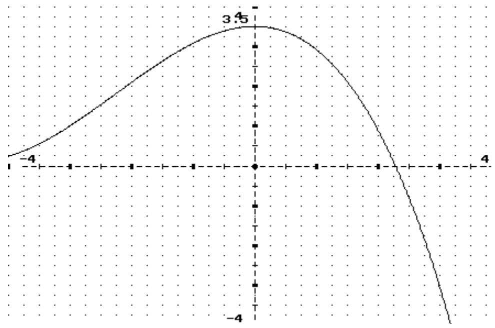
The substitution of these coordinates into (1) would yield the following system of equations,

$$\begin{aligned} -27a + 9b - 3c + d &= 1 \\ 27a + 9b + 3c + d &= -3 \\ 27a - 6b + c &= 1 \\ 27a + 6b + c &= -5 \end{aligned}$$

This system of equations can be solved with the help of IMAGE-*Algebra & Geometry* . The solutions for a, b, c and d yield the cubic,

$$y = -\frac{2}{27}x^3 - \frac{1}{2}x^2 + \frac{7}{2}$$

Submitting this to IMAGE-*Calculus* , we obtain the following graph,



IMAGE|Function|f(x): f(x)=-2x^3/27-x^2/2+7/2

The Binomial Theorem

Given $n, r \in W$, and the following definition from Combinatorial mathematics,

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

where $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$, we can state the **Binomial Theorem** as,

$$\begin{aligned}
(p+q)^n &= \sum_{r=0}^{n} \binom{n}{r} p^{n-r} q^r \\
&= \binom{n}{0} p^{n-0} q^0 + \binom{n}{1} p^{n-1} q^1 + \binom{n}{2} p^{n-2} q^2 + \ldots + \binom{n}{n} p^{n-n} q^n \\
&= \frac{n!}{n!0!} p^n + \frac{n!}{(n-1)!1!} p^{n-1} q + \frac{n!}{(n-2)!2!} p^{n-2} q^2 + \ldots + \frac{n!}{0!n!} q^n \\
&= p^n + np^{n-1} q + \frac{n(n-1)}{2} p^{n-2} q^2 + \ldots + q^n
\end{aligned}$$

Examples

a)

$$\begin{aligned}
(3x+2)^4 &= \sum_{r=0}^{4} \binom{4}{r} (3x)^{4-r} 2^r \\
&= \binom{4}{0} (3x)^{4-0} 2^0 + \binom{4}{1} (3x)^{4-1} 2^1 + \binom{4}{2} (3x)^{4-2} 2^2 + \ldots + \binom{4}{4} (3x)^{4-4} 2^4 \\
&= 81x^4 + 216x^3 + 216x^2 + 96x + 16
\end{aligned}$$

b)

$$\begin{aligned}
(x - \frac{1}{x})^3 &= \sum_{r=0}^{3} \binom{3}{r} x^{3-r} (-\frac{1}{x})^r \\
&= \binom{3}{0} x^{3-0} (-\frac{1}{x})^0 + \binom{3}{1} x^{3-1} (-\frac{1}{x})^1 + \binom{3}{2} x^{3-2} (-\frac{1}{x})^2 + \binom{3}{3} x^{3-3} (-\frac{1}{x})^3 \\
&= x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}
\end{aligned}$$

Application: Probability

A Binomial Experiment is one in which we define two outcomes - one deemed a *success* which denote by the letter, p, and the other a *failure*, denoted by q. Tossing a coin is an obvious example. Since each outcome, a head or a tail is equally likely, or will 50% of the time, we make the assignments, $p = q = \frac{1}{2}$. If we perform the experiment n times, the Binomial Theorem can supply us with a complete probability distribution. Consider a coin tossed three times. Here is the probability distribution,

$$\begin{aligned}
(p+q)^3 &= p^3 + 3p^2 q + 3pq^2 + q^3 \\
(\frac{1}{2} + \frac{1}{2})^3 &= \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 \\
&= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}
\end{aligned}$$

Complex Numbers

Attempts to find roots of the equation, $x^2 + 1 = 0$ within the set of real numbers proves fruitless. A new set of numbers was created to satisfy this equation and ones like it, **the complex numbers**, denoted as the set **C**. The set of complex numbers is defined as the set of binomial expressions of the form $a + bi$ in which $a,\ b \in \Re$ and $i = \sqrt{-1}$,

$$C = \{\ a + bi \mid a, b\ \in \Re\}$$

The Complex (Argand) Plane

The Cartesian Plane, $\Re \times \Re$, or $\Re^2$, is suitable for plotting ordered pairs of the form (x, y) where $x, y \in \Re$. The Complex Plane consists of the ordered pairs (x, y) standing for $x + yi$.

The Magnitude of a Complex Number

Given the complex number $z = a + bi$, we define the magnitude of z, written as $|z|$, to be,

$$|z| = \sqrt{a^2 + b^2}$$

This can be easily seen as the distance from z to the origin of the Complex Plane.

The Complex Conjugate

Early in our study of algebra, we encountered the patterned factoring of a difference of squares,

$$a^2 - b^2 = (a - b)(a + b)$$

We defined the factors as a pair of conjugate binomials: same terms, opposite sign. Similarly, the conjugate of the complex number, $z = a + bi$ is denoted and defined as $\bar{z} = a - bi$.

The Arithmetic of Complex Numbers

Given the two complex numbers, $z_1 = a + bi$ and $z_2 = c + di$ we define the four arithmetic operations as follows,

Addition:

$$\begin{aligned} z_1 + z_2 &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i \end{aligned}$$

Subtraction:

$$\begin{aligned} z_1 - z_2 &= (a + bi) - (c + di) \\ &= (a - c) + (b - d)i \end{aligned}$$

Multiplication:

$$\begin{aligned} z_1 \cdot z_2 &= (a + bi) \cdot (c + di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Division: For division, we rely on an approach similar to *rationalizing the denominator* and the complex conjugate,

$$\begin{aligned} \frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\bar{z}_2}{\bar{z}_2} &= \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \\ &= \frac{(ac + bd)}{c^2 + d^2} + \frac{(bc - ad)}{c^2 + d^2}i \end{aligned}$$

The Square Root of a Complex Number

For z^2 we simply perform $z \cdot z$, but what meaning can be given to $\sqrt{z} = \sqrt{a^2 + b^2}$? For this we could determine the equivalence,

$$a + bi = (x + yi)^2 \tag{1}$$

Expanding (1) unfolds as follows,

$$\begin{aligned} a + bi &= (x + yi) \cdot (x + yi) \\ &= (x^2 - y^2) + (2xy)i \end{aligned}$$

Comparison yields the system of equations,

$$\begin{aligned} x^2 - y^2 &= a \\ 2xy &= b \end{aligned}$$

Given any a and b the outcome of this system can be determined.

Examples

Given $z_1 = -5 + 12i$ and $z_2 = 1 + 2i$, observe the following,

$$\begin{aligned} |z_1| &= |-5 + 12i| \\ &= \sqrt{(-5)^2 + (12)^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{-5 + 12i}{1 + 2i} \cdot \frac{1 - 2i}{1 - 2i} \\ &= \frac{(-5 + 24) + (10 + 12)i}{1 + 4} \\ &= -\frac{19}{5} + \frac{22}{5}i \end{aligned}$$

$$\begin{aligned} z_2 i &= (1 + 2i)i \\ &= i + 2i^2 \\ &= -2 + i \end{aligned}$$

$$\sqrt{z_1} = \sqrt{-5 + 12i}$$

We proceed,

$$\begin{aligned} (x + yi)^2 &= -5 + 12i \\ (x^2 - y^2) + (2xy)i &= -5 + 12i \end{aligned}$$

Equating real and imaginary parts leads to $x^2 - y^2 = -5$ and $2xy = 12$. Solving simultaneously unfolds as follows,

$$\begin{aligned} 2xy &= 12 \\ y &= \frac{6}{x} \end{aligned}$$

Substitution yields,

$$\begin{aligned} x^2 - \left(\frac{6}{x}\right)^2 &= -5 \\ x^2 - \frac{36}{x^2} &= -5 \\ x^4 + 5x^2 - 36 &= 0 \\ (x^2 + 9)(x^2 - 4) &= 0 \end{aligned}$$

From the roots, $x = \pm 2$, we determine $y = \pm 3$. Thus $\sqrt{-5 + 12i} = 2 + 3i$ or $\sqrt{-5 + 12i} = -2 - 3i$.

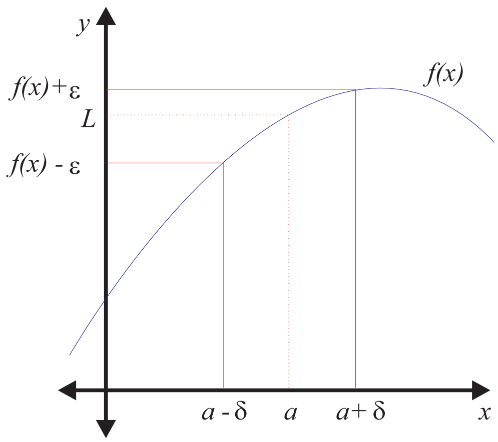
ϵ – δ Definition of a Limit

Few statements in elementary mathematics appear as cryptic as the one defining the limit of a function $f(x)$ at the point $x = a$,

$$\forall\, \epsilon > 0\ \exists\, \delta > 0 \text{ such that } |f(x) - L| < \epsilon \text{ whenever } |x - a| < \delta$$

(**Translation:** *for every epsilon greater than zero, there exists a delta greater than zero such that $f(x)$ lies within epsilon units of the limit L whenever x lies within delta units of a)*

With the aid of a graphic interpretation, the definition is less intimidating.



The definition says that the limit of $f(x)$ at $x = a$ is L if and only if an interval of the domain around a can be found for every interval of the range around L. In other words, to defend a limit result, we must produce a value for δ in response to the challenge of any ϵ.

Example

Should we have to defend the result, $\lim\limits_{x \to 2} 4x - 5 = 3$, we would proceed as follows,

$$\begin{aligned} |(4x - 5) - 3| &< \epsilon \\ |4x - 8| &< \epsilon \\ 4 \cdot |x - 2| &< \epsilon \\ |x - 2| &< \frac{\epsilon}{4} \end{aligned}$$

Since we have determined that δ need only be taken as $\frac{1}{3}$ of any ϵ, the limit result of 3 is substantiated.

Exponent Rules

(a) $x^a x^b = x^{a+b}$

(b) $\frac{x^a}{x^b} = x^{a-b}$

(c) $(xy)^a = x^a y^a$

(d) $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$

(e) $x^{-a} = \frac{1}{x^a}$

(f) $x^0 = 1$

(g) $x^{\frac{1}{n}} = \sqrt[n]{x}$

(h) $x^{\frac{m}{n}} = \sqrt[n]{x^m}$

Logarithm Rules

(a) $\log xy = \log x + \log y$

(b) $\log \frac{x}{y} = \log x - \log y$

(c) $\log x^y = y \cdot \log x$

(d) $\log 1 = 0$

(e) $\log_e x = \ln x$

(f) $\log_y x = \frac{\log_a x}{\log_a y} = \frac{\ln x}{\ln y}$ (*Change of Base* Formula)

(g) $\ln e = 1$

(h) $x^y = e^{y \cdot \ln x}$

Factorial Notation

Locate a button on your calculator with either $n!$ (or $x!$) etched on it (*it could possibly be a second function*). This is read as *n factorial*, and it's mathematical definition is,

Given $n \in W$,

$$n! = \begin{cases} 1 & ; n{=}0 \\ n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1 & ; n{>}0 \end{cases}$$

This is to say that if you enter a 5 and then press $n!$, your calculator will display a result of 120, $(5 \times 4 \times 3 \times 2 \times 1)$.

Examples

a) $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$
b) $\frac{7!}{5!} = \frac{7 \times 6 \times 5!}{5!} = 42$
c) $\frac{8!}{5!3!} = \frac{8 \times 7 \times 6 \times 5!}{5! \times 3 \times 2 \times 1} = \frac{8 \times 7 \times 6}{6} = 56$

Application: Permutations

There is a branch of discrete mathematics dealing with *arrangements* and *combinations* of items. One formula from this area exists which will tell us the number of arrangements that can be made of say, r objects taken from within a larger pool of say, n items. This formula is,

$$P(n, r) = \frac{n!}{(n-r)!} \tag{1}$$

The following exploits this formula. Assume that there are 7 students vying for the four student council positions, *President*, *Vice-President*, *Secretary* and *Treasurer*. How many different ways could the election turn out?

We are faced are with the problem of picking 4 people from a pool of 7. Using (1) above we find,

$$P(7, 4) = \frac{7!}{(7-4)!} = \frac{7 \times 6 \times 5 \times 4 \times 3!}{3!} = 7 \times 6 \times 5 \times 4 = 840$$

Application: Combinations

In the previous example, we say order is important. By this we mean that if after the election the elected Secretary and Treasurer exchanged roles, we would have had a different arrangement! With *combinations* however, order is not important, and hence any exchange of duties from the four elected would not be taken as a new combination. For combinations, we modify (1) to obtain,

$$C(n, r) = \binom{n}{r} = \frac{n!}{(n-r)!r!} \tag{2}$$

Using our election example one last time, asking how many combinations of 4 people can be made from a pool 7 of is like asking how many different ways could a team of 4 students be made, without regard to which role they play. For this we use (2) and find,

$$C(7, 4) = \binom{7}{4} = \frac{7!}{(7-4)!4!} = \frac{7 \times 6 \times 5 \times 4!}{3!4!} = 7 \times 5 = 35$$

More Examples

a) $\dfrac{14!}{13!} = \dfrac{14 \times 13!}{13!} = 14$

b) $\dfrac{52!}{51!} = \dfrac{52 \times 51!}{51!} = 52$

c) $\dfrac{101!}{99!} = \dfrac{101 \times 100 \times 99!}{99!} = 101 \times 100 = 10\,100$

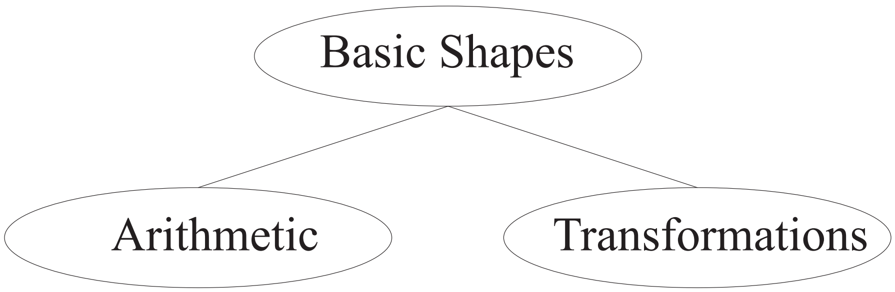
d) $20 \times 19! = 20!$

e) $90 \times 8! = 10 \times 9 \times 8! = 10!$

f) $30 \times 4! = 6 \times 5 \times 4! = 6!$

Graphing Fundamentals

The ability to generate the graph of a function is a useful skill to master. Indeed, the burden of having to find a solution to a problem can be lessened significantly if one can express the context visually. This study introduces very little in the way of new material, instead it will squeeze greater understanding from the concepts you currently possess. The key to rapid, successful graphing is the appreciation of the inseparable links between *algebra* and *geometry*. The sincere student should find these pages well worth the effort.



Sketching functions requires little more than a firm grasp of the three areas depicted above. We will start by reviewing *Basic Shapes*.

Basic Shapes

This section will identify a number of fundamental families of functions. Your responses to the questions and the corresponding graphic should be committed to memory.

Constant Functions, $f(x) = k, \quad k \in \Re$

The graph of a constant function is a horizontal line. With a straight edge, sketch and label the following constant functions,

(a) $f(x) = 0$

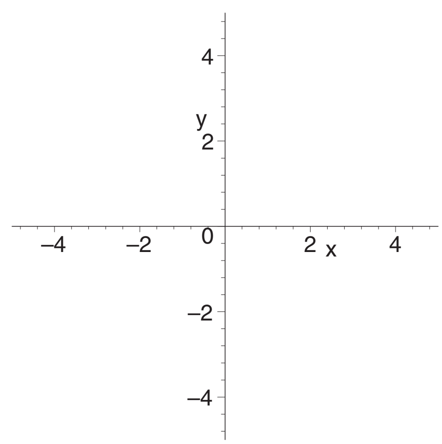
(b) $g(x) = 2$

(c) $h(x) = -1$

(d) State the domain of each function.

(e) State the range of each function.

(f) Identify any symmetry observed for this family.



Power Functions, $f(x) = x^n, \quad n \in N$

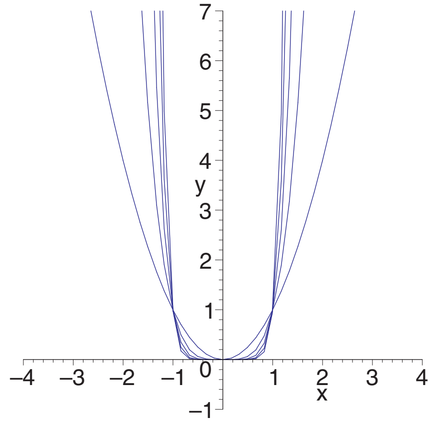
(a) Identify the common symmetrical nature of the functions depicted in the adjacent graphic.

(b) Can you suggest and label the equation of each?

(c) State the common domain and range of each member of this family,

$D = \{x \in \Re|$

$R = \{y \in \Re|$



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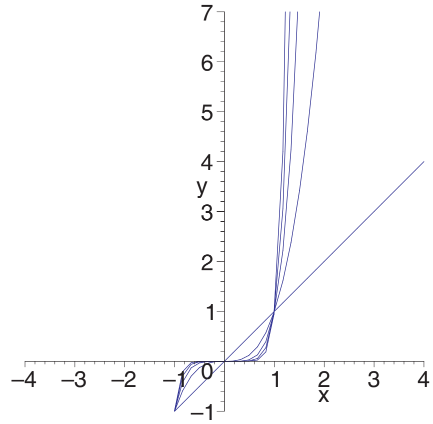

Trigonometric Functions, $f(x) = \sin x, \quad g(x) = \cos x$

(a) Identify the common symmetrical nature of the functions depicted in the adjacent graphic.

(b) Can you suggest and label the equation of each?

(c) State the common domain and range of each member of this family,

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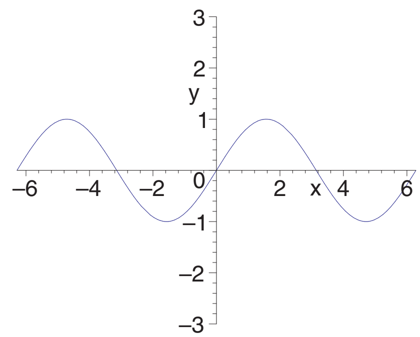

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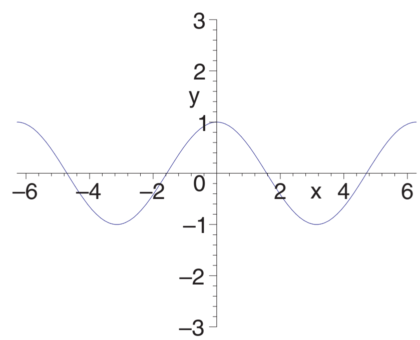


QuickSketch

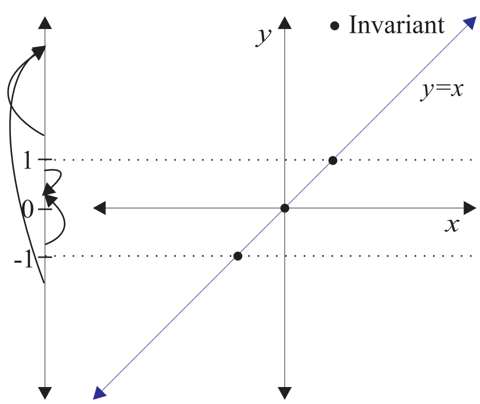
A competent mathematician must be a skilled geometer. Whereas techniques in algebra provide you with the micro precision required to defend the accuracy of your results to an arbitrary number of decimal places, the purpose of geometry is to provide you with the macro or global insight into a concept.

The foundation of the geometric method rests with a sound understanding of fundamental algebraic operations.

This worksheet has been prepared in the hope of pointing you in the right direction.

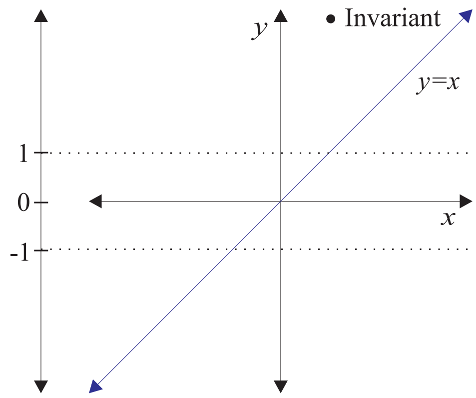
Square: $f(x) = x^2$

One of the first operations you learn in mathematics is that of squaring. The number line presented below crudely summarizes this algebraic transformation. Note any value(s) that do not change under the squaring transformation (*invariant point*). Apply this summary across the domain of $f(x) = x$ to produce the graph of $f(x) = x^2$,



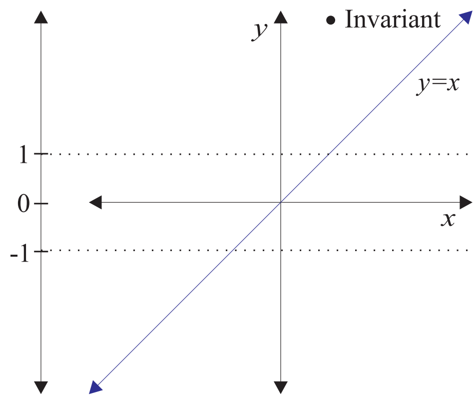
Cube: $f(x) = x^3$

Use the number line presented at the left below to summarize the general results for cubing real numbers (*note any invariant points*). Apply this summary across the domain of $f(x) = x$ to produce the graph of $f(x) = x^3$,



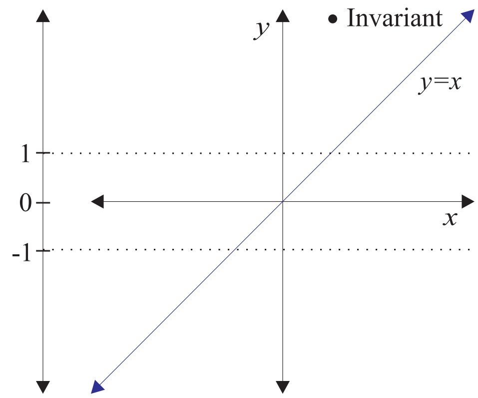
Square Root: $f(x) = \sqrt{x}$

The inverse operation to squaring is taking the square root. Summarize the algebraic effects of taking the square root of a real number on the number line and apply these transformations across the domain of $f(x) = x$ to produce the graph of $f(x) = \sqrt{x}$,



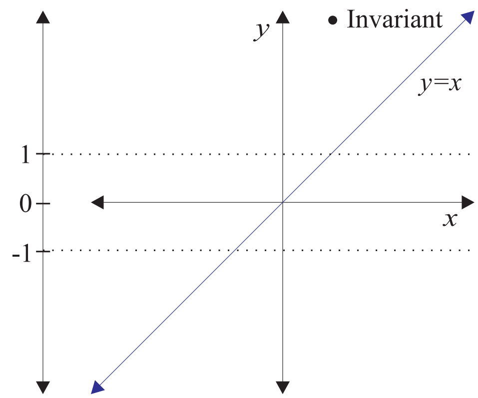
Cube Root: $f(x) = \sqrt[3]{x}$

Use the number line presented at the left below to summarize the general results for taking the cube root of real numbers. Apply this summary across the domain of $f(x) = x$ to produce the graph of $f(x) = \sqrt[3]{x}$,



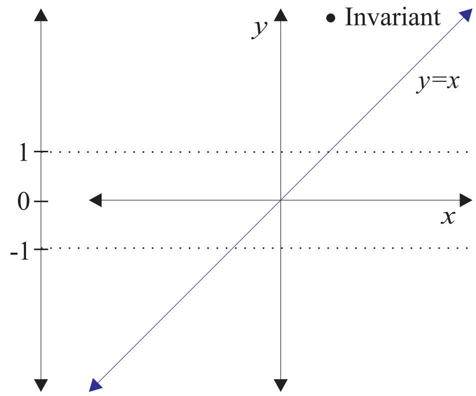
Absolute Value: $f(x) = |\, x \,|$

One of the simplest algebraic transformations involves the absolute value of a number. The result is simply the distance the number is from 0 (*distance can't be negative*). Summarize the algebraic effects of taking the absolute value of a real number and apply these transformations across the domain of $f(x) = x$ to produce the graph of $f(x) = |\, x \,|$,



Reciprocal: $f(x) = \dfrac{1}{x}$

Use the number line presented at the left below to summarize the general results for taking the reciprocal of a real number. Apply this summary across the domain of $f(x) = x$ to produce the graph of $f(x) = \frac{1}{x}$,

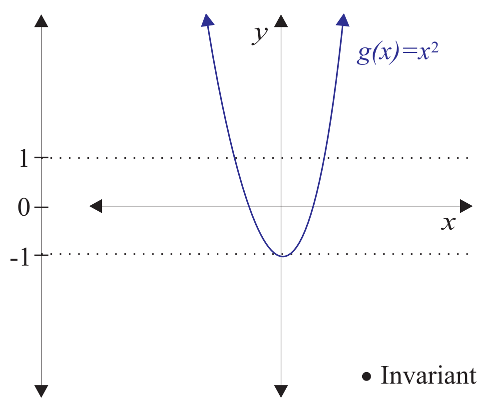


Example 1: $f(x) = \dfrac{1}{x^2 - 1}$

The graph of this function should be seen as applying the reciprocal transformation to the function,

$$g(x) = x^2 - 1$$

Recall the details of the reciprocal transformation on the number line left below to produce the graph of $f(x) = \frac{1}{x^2-1}$,

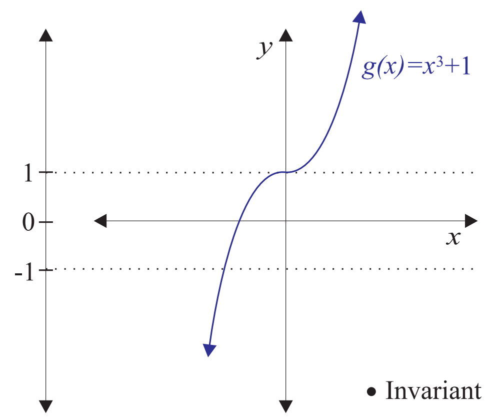


Example 2: $f(x) = |\, x^3 + 1 \,|$

The graph of this function should be seen as applying the absolute value transformation to the function,

$$g(x) = x^3 + 1$$

. Recall the details of the absolute value transformation on the number line left below to produce the graph of $f(x) = |\, x^3 + 1 \,|$,

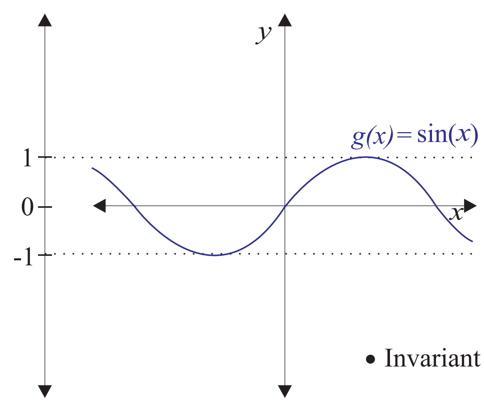


Example 3: $f(x) = (\sin x)^2$

The graph of this function should be seen as applying the squaring transformation to the function,

$$g(x) = \sin x$$

Recall the details of the squaring transformation on the number line left below to produce the graph of $f(x) = (\sin x)^2$,



Exercises

1. Develop the graphs of the given functions using the techniques presented.

   a) $f(x) = \dfrac{1}{2x - 1}$

   b) $f(x) = \sqrt[3]{x^2} = x^{\frac{2}{3}}$

   c) $f(x) = \dfrac{1}{x^2}$

   d) $f(x) = \dfrac{1}{\sqrt{x - 3}}$

   e) $f(x) = \sqrt{4 - x^2}$

   f) $f(x) = \dfrac{1}{\sin x}$

2. State the domain and range of each of the functions in 1.

Indefinite Integration: An Interesting Evaluation

Using the definition of absolute value,

$$|x| = \sqrt{x^2}$$

we recall the derivative of $y = |x|,\ x \neq 0$ as,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d|x|}{dx} \\
&= \frac{d\sqrt{x^2}}{dx} \\
&= \frac{d\sqrt{x^2}}{d\,x^2} \cdot \frac{d\,x^2}{d\,x} \\
&= \frac{1}{2\sqrt{x^2}} \cdot 2x \\
&= \frac{x}{\sqrt{x^2}} \\
&= \frac{x}{|x|}
\end{aligned}$$

The integral $\int |x|\ dx$, can be expressed as $\int \sqrt{x^2}\ dx$ and we can proceed with *Integration by Parts* by letting $u = \sqrt{x^2}, \quad dv = dx$

$$\begin{aligned}
\int |x|\ dx &= x\sqrt{x^2} - \int \frac{2x^2}{2\sqrt{x^2}}\ dx \\
&= x|x| - \int \frac{x^2}{\sqrt{x^2}}\ dx \\
&= x|x| - \int \sqrt{x^2}\ dx \\
&= x|x| - \int |x|\ dx
\end{aligned}$$

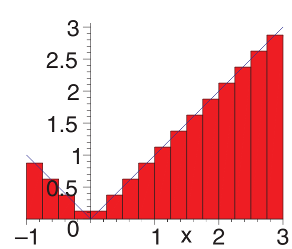
Adding $\int |x|\ dx$ to both sides yields,

$$\begin{aligned}
2\int |x|\ dx &= x|x| \\
&= \frac{x|x|}{2} + C
\end{aligned}$$

Demonstration

Determine the area under the function, $f(x) = |x|$, on the partition, $[-1, 3]$.

$$\begin{aligned}
A = \int_{-1}^{3} |x|\ dx &= \left[\frac{x|x|}{2}\right]_{-1}^{3} \\
&= \frac{3|3|}{2} - \frac{-1|-1|}{2} \\
&= \frac{9}{2} + \frac{1}{2} \\
&= 5u^2
\end{aligned}$$



Applications of the Definite Integral: Arc Length

A formula introduced in intermediate mathematics,

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} \tag{1}$$

yields the straight line distance, d, between two points. In addition to straight segments, we can also determine the length of an arc of a circle, ℓ, with the aid of,

$$\frac{\theta}{2\pi} = \frac{a}{\pi r^2} = \frac{\ell}{2\pi r}$$

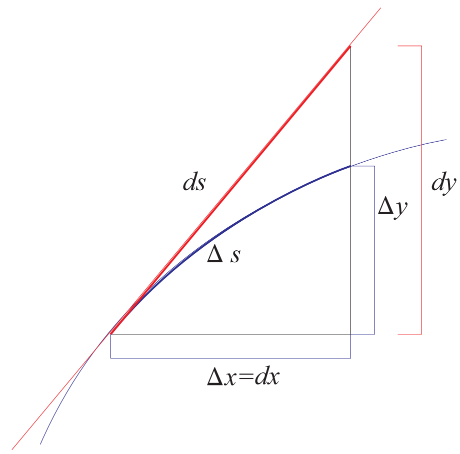
However, we must turn to the definite integral for a general method of quantifying non-linear measures.

In your earlier study of limits, you were asked to conceptualize the circle as the limit of a n-sided regular polygon where $n \to \infty$. In particular, the circumference of the circle arises from the sum of the infinite number of infinitesimally short sides of the regular polygon. We can adapt this concept to the problem of finding the length of an arc of a function.

From our study of differentials, we are aware that Δy can be approximated by dy where

$$dy = f'(x)\, dx \tag{2}$$

Using a Riemann Sum approach, we can incorporate concepts (1) and (2) into the development of an integral formula for the length, s, of an arc of the function, $f(x)$.



We must first develop ds as an expression for Δs. The Pythagorean relationship,

$$(\Delta s)^2 \approx (\Delta x)^2 + (\Delta y)^2$$

can be approximated through the use of differentials,

$$(ds)^2 = (dx)^2 + (dy)^2$$

substituting (2), we simplify as follows,

$$\begin{aligned}(ds)^2 &= (dx)^2 + (f'(x)\, dx)^2 \\ (ds)^2 &= (1 + [f'(x)]^2)\,(dx)^2 \\ ds &= \sqrt{1 + [f'(x)]^2}\, dx\end{aligned}$$

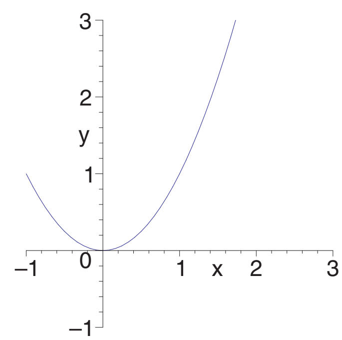
The formula for the length of an arc of the function, $f(x)$, over the continuous closed interval $[a, b]$ is completed as follows,

$$\begin{aligned} s &= \lim_{n\to\infty} s_n \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta s \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x)]^2}\, dx \\ &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \end{aligned}$$

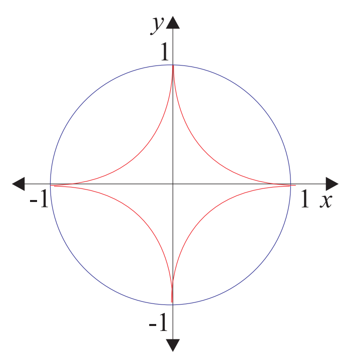
Questions

1. Confirm the formula for the circumference of a circle through the use of the definite integral.

2. Determine the length of the arc of the parabola $y = \frac{1}{2}x^2$ over the interval [0,2]. (Ans. $\approx$ 2.9579 units)



3. Determine the total length of the *hypocycloid of four cusps* whose equation can be given by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$. (Ans. 6 units)



π

The set of real numbers, $\Re$, is a union of the rational numbers, Q, and the irrational numbers, $\bar{Q}$,

$$\Re = Q \cup \bar{Q}$$

Your first investigation of the qualitative properties of the circle produced a formula for it's circumference, C,

$$C = 2\pi r$$

which can be reexpressed as,

$$\pi = \frac{C}{2r} = \frac{C}{d}$$

From this point on, we recognize π as the constant ratio of the circumference of a circle to it's diameter. Since $\pi \in \bar{Q}$, we can not know an exact value for this constant. A crude approximation of this constant is taken as 3.14. A slightly better expression is fraction, $\frac{22}{7}$. An even better rational approximation is $\frac{355}{113}$. All sorts of interesting computational techniques have been devised, including this one handed to me by Dr. Skalinski,

$$\frac{2 \times 2 \times 4 \times 4 \times 6 \times 6 \ldots}{1 \times 3 \times 3 \times 5 \times 5 \times 7 \ldots} \approx \frac{\pi}{2}$$

Using series, it is possible to obtain as close an approximation to the value of π as you wish. Many series exist, two of which are presented below. These can be evaluated by hand, but if you know a computer language, it is fairly painless and quite rewarding to write a program to sum the terms of each series.

Series A

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Series B

$$\pi = \sum_{n=0}^{\infty} 16^{-n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right)$$

Each successive term of these infinite series is smaller than it's predecessor. We can interpret this to mean that the difference between our approximate value and the theoretical values of π is shrinking. One could deem one series superior to another based on how fast this difference shrinks. Which of the two series presented is superior according to this criteria?

Factoring: Differences and Sums

The following results are well known,

$$a - b = (a - b)(1) \tag{1}$$

and,

$$a^2 - b^2 = (a - b)(a + b) \tag{2}$$

The question is now asked, *"Can we extend these results to factor $a^n - b^n$ for $n > 2,\ n \in \Re$?"*. Let us examine the case for $n = 3$. For $n = 1$ and $n = 2$, we observe the first factor to be $a - b$. We could assume this pattern continues, and test our assumption through division: $(a^3 - b^3) \div (a - b)$. Our assumption is correct, and we discover,

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \tag{3}$$

Putting (1), (2) and (3) together reveals a pattern that can easily be verified,

$$\begin{aligned} a - b &= (a - b)(1) \\ a^2 - b^2 &= (a - b)(a + b) \\ a^3 - b^3 &= (a - b)(a^2 + ab + b^2) \\ a^4 - b^4 &= (a - b)(a^3 + a^2 b + ab^2 + b^3) \end{aligned}$$

To confirm whether the previous factoring is correct, we would simply multiply. Generalizing this result produces,

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1}) = (a - b)\sum_{r=0}^{n-1} a^{n-r-1}b^r \tag{4}$$

We shall now examine the factoring of sums. In searching for potential factors of the expression, x^2+1, there are many avenues one could take. My personal favourite is to think geometrically. In the study, *Factors and Roots*, we were reminded of the inseparable link between *algebra* and *geometry*. Bringing this to bear on our immediate problem, we can easily visualize the graph of the function $f(x) = x^2 + 1$. Since the graph of the function fails to make contact with the x-axis, we conclude there are <u>no roots</u>, hence, <u>no linear factors</u>. To this point, we have,

$$x + 1 = (x + 1)(1) \tag{5}$$

$$x^2 + 1 \quad no\ factors \tag{6}$$

For $n = 3$, we could visualize the function, $f(x) = x^3 + 1$. Our mind's eye spots one (*and only one*) x-intercept, leading to a single linear factor. It would not be too great a leap to speculate that the root might be -1, so we could hazard a guess that one factor of $x^3 + 1$ might be $x + 1$. Division yields the following,

$$x^3 + 1 = (x + 1)(x^2 - x + 1) \tag{7}$$

A few seconds of mental imagery rules out linear factors for the expression, $x^4 + 1$. The same logical approach we pursued to arrive at (7) can be used to discover factors in the case of $n = 5$,

$$x^5 + 1 = (x + 1)(x^4 - x^3 + x^2 - x + 1) \tag{8}$$

We can now put (5), (6), (7) and (8) together to build a general result for factoring a sum for *odd n*, that can be verified,

$$a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots - ab^{n-2} + b^{n-1}) = (a + b)\sum_{r=0}^{n-1} (-1)^r a^{n-r-1}b^r \tag{9}$$

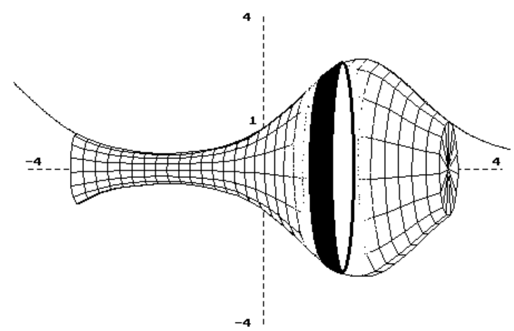
Surface Area of a Solid of Revolution

Consider a region bounded by the function, $f(x)$, the x-axis and the lines $x = a$ and $x = b$. Sweeping this area around the x-axis generates an object figure known appropriately as a *solid of revolution*. For such objects we have previously explored the formula for the volume through the use of the definite integral,

$$V = \pi \int_a^b [f(x)]^2 \, dx$$

We now turn to the development of a formula for the *surface area* of the solid of revolution.

From a Riemann Sum perspective, the formula for volume was developed by considering the sum of an infinite number of cylindrical volumes whose base radii were dictated by $f(x)$. For surface area, we return to the notion of the varying cylindrical forms but consider instead their lateral areas.



The lateral area of a cylinder can be treated as a rectangle with dimensions $2\pi r \times \Delta x$ which can adjusted to $2\pi f(x) \times ds$ where ds is our expression for a slice of arc length from a previous investigation,

$$ds = \sqrt{1 + [f'(x)]^2} \, dx$$

Our formula for *surface area* is built as follows,

$$\begin{aligned} SA &= \lim_{n\to\infty} sa_n \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta sa \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \, f(x)\sqrt{1 + [f'(x)]^2} \, dx \\ &= 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2} \, dx \end{aligned}$$

Questions

1. Confirm the formula for the surface area of a sphere through the use of the definite integral.

2. Find the surface area of a zone of altitude h of a sphere of radius r. (*A zone is the portion of a sphere included between two parallel planes; the altitude of a zone is the distance between the two planes*)

3. Confirm the formula for the surface area of a right circular cone.

The Area of a Triangle

Numerous techniques exist for determining the area of a triangle. Four such approaches are reviewed in this document.

1. Arithmetic
2. Trigonometry
3. Heron's Formula
4. Analytic Geometry

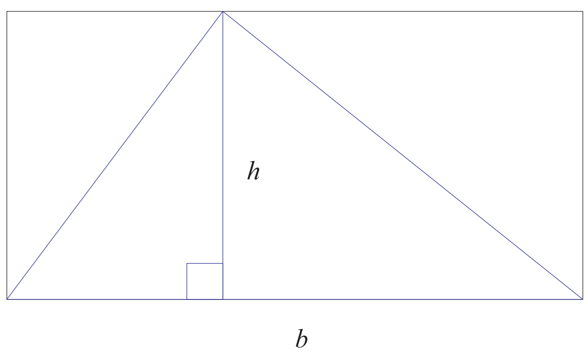
The best method depends on the information provided.

1. Arithmetic

To each side of a triangle, an altitude can be determined. Letting the side be the base, b, and its altitude be h, the area, A, of the triangle can be expressed as

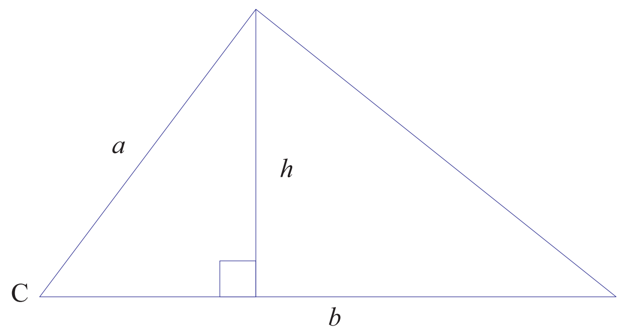
$$A = \frac{1}{2}bh \tag{1}$$

The rationale for (1) is straightforward. Any triangle can define a rectangle containing it whose area is bh. The triangle consumes one-half of the area of the rectangle.



2. Trigonometry

Given the lengths of two sides of a triangle and the measure of the contained angle, a height can be determined using trigonometry.



In the triangle above we have,

$$\sin C = \frac{h}{a}$$
$$h = a \sin C \tag{2}$$

Substituting (2) into (1) yields a convenient formula for the area, A given two sides and the contained angle

$$A = \frac{1}{2} ab \sin C \tag{3}$$

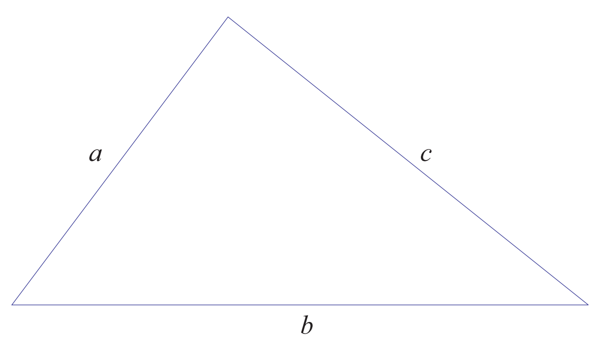
3. Heron's Formula

Should the lengths of the three sides of a triangle be readily known, Heron's formula supplying a value for the area, A, is handy,

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{4}$$

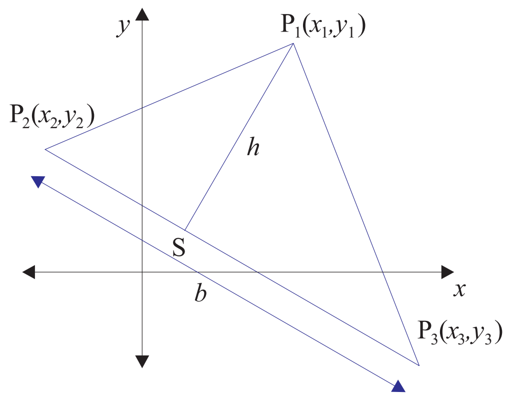
where $a,\ b$, and c are the lengths of the sides of the triangle and s is the semi-perimeter.

$$s = \frac{1}{2}(a+b+c) \tag{5}$$



4. Analytic Geometry

The Arithmetic approach examined earlier yielded an area of $\frac{1}{2}bh$ for a side taken as the base, b, and the corresponding height, h. This formula can be adapted to to develop an approach using coordinate geometry.



Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$ be the vertices of any triangle.

Let $P_3P_2 = b$ be the base of the triangle.

Let $P_1S = h$ be the height of the triangle.

We have

$$b = \sqrt{(\Delta x)^2 + (\Delta y)^2} \tag{6}$$

where

$$\Delta x = x_2 - x_3 \text{ and } \Delta y = y_2 - y_3$$

We can now develop and expression for h using the formula for the distance, d, from a point, (x_1, y_1) to a line, $Ax + By + C = 0$,

$$d = \frac{|\ Ax_1 + By_1 + C\ |}{\sqrt{A^2 + B^2}} \tag{7}$$

The equation of the line through P_2 and P_3 can be developed as follows,

$$\begin{aligned} \frac{y - y_2}{x - x_3} &= \frac{\Delta y}{\Delta x} \\ (\Delta y)x - (\Delta y)x_3 &= (\Delta x)y - (\Delta x)y_3 \\ (\Delta y)x - (\Delta x)y + (\Delta x)y_3 - (\Delta y)x_3 &= 0 \end{aligned} \tag{8}$$

Substituting (6) and (8) into (7) yields an expression for h.

$$h = \frac{|\ (\Delta y)x_1 - (\Delta x)y_1 + (\Delta x)y_3 - (\Delta y)x_3\ |}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

The area of $\triangle P_1P_2P_3$ can expressed as

$$\begin{aligned} |P_1P_2P_3| &= \frac{1}{2}bh \\ &= \frac{1}{2} \times \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{1} \times \frac{|\ (\Delta y)x_1 - (\Delta x)y_1 + (\Delta x)y_3 - (\Delta y)x_3\ |}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \\ &= \frac{1}{2}|\ (\Delta y)x_1 - (\Delta x)y_1 + (\Delta x)y_3 - (\Delta y)x_3\ | \end{aligned} \tag{9}$$

Substituting $x_2 - x_3$ for Δx and $y_2 - y_3$ for Δy in (9) yields a final formula for the area of our triangle.

$$\begin{aligned}|P_1P_2P_3| &= \frac{1}{2}|(y_2 - y_3)x_1 - (x_2 - x_3)y_1 + (x_2 - x_3)y_3 - (y_2 - y_3)x_3| \\ &= \frac{1}{2}|x_1y_2 - x_1y_3 - x_2y_1 - x_3y_1 + x_2y_3 - x_3y_3 - x_3y_2 + x_3y_3| \\ &= \frac{1}{2}|x_1y_2 + x_2y_3 + x_3y_1 - (x_2y_1 + x_3y_2 - x_1y_3| \qquad (10)\end{aligned}$$

The following pattern can simplify the remembrance of the above formula.

Points	Down Products	Up Products
(x_1, y_1)	$x_1 \quad y_1$	$x_1 \quad y_1$
(x_2, y_2)	$x_2 \quad y_2$	$x_2 \quad y_2$
(x_3, y_3)	$x_3 \quad y_3$	$x_3 \quad y_3$
(x_1, y_1)	$x_1 \quad y_1$	$x_1 \quad y_1$

Sum of the Down Products $= x_1y_1 + x_2y_3 + x_3y_1$

Sum of the UpProducts $= x_2y_1 + x_3y_2 + x_1y_3$

$$\text{Area} = \frac{1}{2}|(\textit{Sum of the Down Products}) - (\textit{Sum of the Up Products})|$$

Example

Find the area of ΔABC with coordinates A$(3, 2)$, B$(-1, -1)$ and C$(2, -3)$.

$$\begin{aligned}\text{Area } \Delta\text{ABC} &= \frac{1}{2}|(-3 + 3 + 4) - (-9 - 2 - 2)| \\ &= \frac{1}{2}|4 + 13| \\ &= 8\frac{1}{2}\end{aligned}$$

The area of the triangle is $8\frac{1}{2}$ square units.